Optimal Distributed Decision Making using Team Theory

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A group of individuals with different information obtained through their respective observations of the environment, who take decision about something different but having common goal and preferences and receive a common reward as a joint result of all the individual decisions.
The Pioneers
Marschak and Radner

Jacob Marschak

Roy Radner
The Team
Origins

- Theory of Teams
  - special case of theory of games
  - basic to theory of optimal organization
- Organization: for a given state of world, the individuals
  1. make different observations and get different information
  2. take different decisions based on their observations
  3. have different preferences in terms of tastes, believes, etc.

‘Team is an organization with members having same preferences’ – Marschak

- idea of ‘team’ and ‘decision makers’
Individual members of a team differ \( w.r.t \) their
- information
- possibilities of actions
but agree in
- goal and/or preferences
There is uncertainty about
- state of the world
- other members’ actions
Notations

- $s$ state of world; a description of environment so complete that if true and known, the consequences of every action would be known

- $y_i$ information; signal received through the process of observation
  \[ y_i = \eta_i(s) \]

- $d_i$ decision; action taken based on the information
  \[ d_i = \delta_i(y_i) = \delta_i[\eta_i(s)] \]
For a team $T = 1, 2, \ldots, N$

\[
\eta = \begin{bmatrix}
\eta_1 \\
\eta_2 \\
\vdots \\
\eta_N 
\end{bmatrix}, \quad \delta = \begin{bmatrix}
\delta_1 \\
\delta_2 \\
\vdots \\
\delta_N 
\end{bmatrix}, \quad d = \begin{bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_N 
\end{bmatrix}
\]
Payoff is a function of state and team decision

\[ \omega(s, d) \]

Objective: Maximize the expectation of payoff
(s is a random variable)

\[ E[\omega(s, d)] = \sum_{s \in S} \phi(s)\omega(s, \delta[\eta(s)]) \]

Goal: Choose the best team decision function \( \delta \) to maximize expected payoff for given information structure
If $\hat{\delta}$ is an optimal solution then it is necessarily person-by-person-optimal (pbpo)

If the expected payoff cannot be improved by changing only one member’s decision function then such a decision is person-by-person-optimal or coordinate-wise optimal.
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Does pbp optimality imply a global optimum?
Example

\[ \omega(d_1, d_2) = \min\{-d_1^2 - (d_2 - 1)^2, (d_1 - 1)^2 - d_2\} \]
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When does pbpo becomes sufficient for global optimality?
**Theorem 1 (Radner, 1962)**

If the team payoff function $\omega(s, d)$ is concave and differentiable in decision variables $d_i$ for every state $s$, then person-by-person-optimal solution global optimum.

Proof:
Let $\hat{\delta}$ be the pbpo solution.
Define

$$\mathbf{k} = (k_1, \ldots, k_N)$$
$$\beta_i(s) = \delta_i(s) - \hat{\delta}_i(s), \quad i = 1, \ldots, N$$
$$f(s, \mathbf{k}) = \omega[s, \hat{\delta}_1(s) + k_1 \beta_1(s), \ldots, \hat{\delta}_N(s) + k_N \beta_N(s)]$$
Looking for Optimal Solutions

The expectation of cost function is then

\[ F(k) = \sum_{s \in S} \phi(s)f(s, k) \]

For a \( k \) of the form \( k = (0, 0, \ldots, k_i, \ldots, 0) \)

\[ F(k) \leq F(0) \]

This along with differentiability implies

\[ \left[ \frac{\partial F(k)}{\partial k_i} \right]_{k=0} = 0, \quad \forall i = 1, \ldots, N \]

Now since function \( F \) is also concave, \( k = 0 \) is globally optimal.
Teams with quadratic payoff function

$$\omega(s,d) = \lambda(s) + 2\mu^T(s)d - d^T\nu(s)d$$

**Theorem 2 (Radner, 1962)**

If $\nu$ is positive definite and symmetric, then for a given information $y_i$ the optimal decision functions are uniquely determined by

$$\delta_i(y_i)E(\nu_{ii}|y_i) + \sum_{j \neq i} E(\delta_j \nu_{ij}|y_i) = E(\nu_i|y_i), \quad i = 1, \ldots, N$$
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Is it possible to get the form of decision function?
Optimal Teams
Quadratic Payoff Function

\[ \omega(s, d) = \lambda(s) + 2\mu^T(s)d - d^T \nu(s)d \]

**Theorem 3 (Radner, 1962)**

If \( \nu \) is constant and involved random variables are normally distributed with \( E(y_i) = 0 \) and \( \text{Var}(y_i) = 1 \), then the optimal decision is given by

\[ \delta_i(y_i) = b_i y_i + c_i \]

where \( b_i \) and \( c_i \) are solutions of linear system

\[
\sum_j \nu_{ij} \text{Cov}(y_i, y_j) b_j = \text{Cov}(y_i, \mu_i) \quad \text{and} \quad \sum_j \nu_{ij} c_j = E(\mu_i)
\]
Optimal Teams
Quadratic Payoff Function

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Optimal decision is linear in information variables
Payoff function is concave and polyhedral

Team decision problem equivalent to linear programming problem in the ‘decision function space’

Payoff function linear in decision variable (Linear Team)

Decision problem already in linear programming form

Linear payoff function

\[ \omega(s, d) = \sum_i f_i(s)d_i \]

Objective is to maximize

\[ E(\omega(s, d)) = \sum_s \sum_i f_i(s)d_i\phi(s) \]

subject to

\[ \sum_i g_i(s)d_i \leq c \]
Payoff function is concave and polyhedral
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Linear Programming under uncertainty
Task Allocation using Team Theory

The Problem

A distributed decision problem with team theoretic flavor

Individual members of a team differ \( w.r.t \) their

- information
- possibilities of actions

but agree in

- goal and/or preferences

There is uncertainty about

- state of the world
- other members’ actions
Task Allocation
Problem Details

- $N$ UAVs and $M$ targets
- State

\[ s = \{ \text{UAV position, target position, target value} \} \]

- Target value, $r \in \{0, 0.5, 1\}$
- Decision of $i^{th}$ UAV

\[ d_i = \{ d_{i1}, d_{i2}, \ldots, d_{i(m_i+1)} \} \]

\[ d_{ij} = \begin{cases} 
1 & \text{if UAV } i \text{ attacks target } j \\
0 & \text{otherwise}
\end{cases} \]

$m_i$ is number of targets in sensor range
$d_{i(m_i+1)}$ is search task for UAV $i$
Task Allocation
Problem Formulation

- Payoff function

\[ \omega = \sum_{i=1}^{n_i} \sum_{j=1}^{m_i+1} C_{ij} d_{ij} \]

- Constraints

1. UAV can execute only one task at a time

\[ \sum_{j=1}^{m_i+1} d_{ij} = 1, \quad \forall i = 1, \ldots, n_i \]

2. A target is not attacked by more than one UAV

\[ \sum_{i=1}^{n_i} d_{ij} \leq 1, \quad \forall j = 1, \ldots, m_i \]
Modeling the cost function

- Cost of search = \( \frac{\text{time left in mission}}{\text{total time of mission}} \)

  promote search in the beginning of mission

- Cost of attack = target value \( \times w - \frac{\text{time to reach target}}{\text{total time of mission}} \)

  attack closer and higher value targets
Task Allocation
Modeling

Accounting for other members’ decisions

\[ \omega = \sum_{i=1}^{n_i} \sum_{j=1}^{m_i+1} C_{ij} d_{ij} \]

Virtual targets

Virtual UAVs
Task Allocation
Team Theoretic Formulation

Objective: Solve the Linear Programming problem

\[
\max_d E \left( \sum_{i,j} C_{ij} d_{ij} \right)
\]

subject to

\[
\sum_j d_{ij} = 1, \quad \sum_i d_{ij} \leq 1
\]
Observation depends on other decision makers’ decisions

Optimal strategy exist only under assumptions of

1. Linear Quadratic Gaussian problem
2. Partially Nested information structure

Ex: Source Coding problem

Summing Up

- Definition of team
- Mathematical framework for team problems
  - state
  - information structure
  - decision function
  - payoff function
- Optimal team decisions
  - concavity and differentiability
  - quadratic and polyhedral payoff
- Task allocation example
- Dynamic teams
In a Nutshell

Take home message

Team theory provides a suitable framework for the correct statement of an informally decentralized optimization problem and enables one to determine analytically the optimal strategies in a few cases.