Almost all of the results presented are taken from

Reference

Consensus over networks and the effect of network topology on stability of multi-agent systems

Joel George

September 21, 2007
Representation of Multi-Agent System
Multi-Agent Systems

Representation of Multi-Agent System

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Background Theory Consensus Protocol Convergence Analysis Formation Control Applications

Graph Theory
Adjacency Matrix

Graph Representation

Matrix Representation

Adjacency Matrix

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Graph Theory

Degree Matrix

Graph Representation

Matrix Representation

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 3 \\
\end{bmatrix}
\]
Laplacian Matrix

\[ L = D - A \]

- Has zero eigenvalue corresponding to eigenvector \( \mathbf{1} \)
- Is positive semi-definite (\( x^T L x > 0 \))
Graph Theory
Graph Connectivity and Fiedler Eigenvalue

Connected Graph

\[ G \]
Graph Theory
Graph Connectivity and Fiedler Eigenvalue

Disconnected Graph

\[ \mathcal{G}_1 \]

\[ \mathcal{G}_2 \]

\[ \mathcal{G} \]
Disconnected Graph

\[ L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \]
Graph Theory
Graph Connectivity and Fiedler Eigenvalue

Disconnected Graph

\[ L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \]

Eigenvalues of \( L \)

\[ 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2\Delta \]

\[ \Delta = \max_i d_i \]
Graph Theory
Graph Connectivity and Fiedler Eigenvalue

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\[ L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \]

Eigenvalues of $L$

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Fiedler Eigenvalue

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Consensus Protocol

The Algorithm

\[
\dot{x}_i(t) = \sum_{j \in N_i} (x_j(t) - x_i(t))
\]

\[
\dot{x} = -Lx
\]
The Consensus Protocol

The Algorithm

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Consensus Protocol

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\[ \dot{x} = -Lx \]
Adjacency matrix, $A = [a_{ij}]$ with $a_{ij} = a_{ji}$

$$
\dot{x}_i(t) = \sum_{j \in N_i} a_{ij} (x_j(t) - x_i(t))
$$

$$\dot{x} = -Lx$$

where

$$
L = D - A
$$

$$
D = \text{diag}(d_1, d_2, \ldots, d_n)
$$

$$
d_i = \sum_{i \neq j} a_{ij}
$$
Consensus Protocol

Properties

- Equilibrium of $\dot{x} = -Lx$ is
  \[ x^* = \alpha \mathbf{1} \]

- $\sum_i \dot{x}_i = 0$ is invariant
- Gives average consensus with
  \[ \alpha = \frac{1}{n} \sum_i x_i(0) \]

- Sum-of-Squares Property
  \[ x^T Lx = \frac{1}{2} \sum_{i,j} a_{ij} (x_j - x_i)^2 \]

- Implications for large networks
**Theorem 1**

Let $G$ be a connected graph. Then, the consensus algorithm

$$\dot{x}_i(t) = \sum_{j \in N_i} a_{ij} (x_j(t) - x_i(t))$$

asymptotically solves an average consensus problem for all initial states.
More realistic modeling of networks
More realistic modeling of networks
Lemma 1

Let $G$ be a strongly connected digraph on $n$ nodes. Then, $\text{rank}(L) = n - 1$ and all non-trivial eigenvalues of $L$ have positive real parts.

Remarks:

- Lemma 1 holds if there exists a spanning tree for $G$. 
Lemma 1 implies consensus is reached asymptotically if \( \gamma \) be the left eigenvector of \( L \), \( \gamma^T L = 0 \)

Note: \( y = \gamma^T x \) is invariant as \( \dot{y} = \gamma^T \dot{x} = -\gamma^T L x = 0 \)

\[ \lim_{t \to \infty} y(t) = y(0) \implies \gamma^T (\alpha 1) = \gamma^T x(0) \]

The group decision,

\[ \alpha = \frac{\gamma^T z}{\gamma^T 1} \]
**Definition Balanced Digraph**

A digraph $G$ is called a balanced digraph if

$$\sum_{j \neq i} a_{ij} = \sum_{j \neq i} a_{ji}$$

**Example Balanced Digraph**

$$A = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 \end{bmatrix}$$
Remarks:

1. Balanced digraph has \( \mathbf{1} \) as left eigenvector
2. Average consensus is achieved

The group decision,

\[
\alpha = \frac{1}{n} \sum_{i} x_i(0)
\]
**Theorem 2**

Consider a network of $n$ agents with topology $G$ applying consensus algorithm (A). Suppose $G$ is strongly connected digraph. Let $\gamma$ be the left eigenvector of Laplacian of $G$. Then

i) a consensus is asymptotically reached for all initial states

ii) the algorithm solves consensus problem ($x^* = \alpha \mathbf{1}$) with group decision

$$\alpha = \frac{\gamma^T z}{\gamma^T \mathbf{1}}$$

iii) if the digraph is balanced, an average consensus is asymptotically reached
Discrete Time Consensus Protocol

\[ x_i(k + 1) = x_i(k) + \epsilon \sum_{j \in N_i} a_{ij} (x_j(k) - x_i(k)) \]  

\[ x(k + 1) = Px(k) \]

where

\[ P = I - \epsilon L \]

is called the \textit{Perron} matrix
Definitions:

- A matrix is **nonnegative** if all its entries are greater than or equal to zero.
- A matrix is **irreducible** if its associated graph is strongly connected.
- A nonnegative matrix is called a row (or column) **stochastic** if all of its row-sums (column-sums) are 1.
- An irreducible stochastic matrix is **primitive** if it has only one eigenvalue with maximum modulus.
Lemma 2

Let $G$ be a digraph with $n$ nodes and maximum degree $\Delta = \max_i \left( \sum_{j \neq i} a_{ij} \right)$. Then the Perron matrix $P$ with parameter $\epsilon \in (0, 1/\Delta]$ satisfies the following properties:

i) $P$ is a row stochastic nonnegative matrix with a trivial eigenvalue of 1

ii) All eigenvalues of $P$ are in the unit circle

iii) If $G$ is a balanced digraph then, $P$ is a doubly stochastic matrix

iv) If $G$ is strongly connected and $0 < \epsilon < 1/\Delta$ then $P$ is a primitive matrix
Theorem 3

Consider a network of agents with topology $G$ applying distributed consensus algorithm (B) with $0 < \epsilon < 1/\Delta$. Let $G$ be strongly connected digraph. Then

i) a consensus is asymptotically reached for all initial states

ii) the group decision value is $\alpha = \sum_i w_i x_i(0)$ with $\sum_i w_i = 1$

iii) if digraph is balanced (or $P$ is doubly stochastic), an average-consensus is asymptotically reached.
Disagreement vector
\[ \delta = x - \alpha 1 \]

Disagreement dynamics
\[ \dot{\delta}(t) = -L\delta(t) \]

A valid Lyapunov function
\[ \Phi(\delta) = \delta^T \delta \]
\[ \dot{\Phi}(\delta) = 2\delta^T \dot{\delta} = -\delta^T L\delta < 0 \]
**Theorem 4**

Let $G$ be a balanced digraph with Laplacian $L$ with a symmetric part $L_s = (L + L^T)/2$. Let $\lambda_2 = \lambda_2(L_s)$. Then

$$\delta^T L \delta \geq \lambda_2 \| \delta \|^2$$

**Corollary 1**

A continuous time consensus is globally exponentially reached with a speed that is faster or equal to $\lambda_2 = \lambda_2(L_s)$ for a strongly connected and balanced digraph.
Theorem 5

The algorithm (C) asymptotically solves the average-consensus problem with a uniform one-hop time-delay $\tau$ for all initial states if and only if $0 \leq \tau < \pi / 2\lambda_n$. 

\[
\dot{x}(t) = -Lx(t - \tau)
\]
In real world networks

- Neighbors keep changing
- Connections break and form

Consider the protocol

\[ \dot{x} = -L(G_k)x, \quad G_k \in \Gamma = \{G_1, \ldots, G_m\} \]  \hspace{1cm} (D)

with switching function

\[ s(t) \mapsto k, \quad k \in \{1, \ldots, m\} \]
THEOREM 6

Consider a network of agents applying consensus algorithm in \((D)\) with topologies \(G_k \in \Gamma\). Suppose every graph in \(\Gamma\) is balanced digraph that is strongly connected and let \(\lambda^*_2 = \min_k \lambda_2(G_k)\). Then, for any arbitrary switching signal, the agents asymptotically reaches an average-consensus for all initial states with a speed faster or equal to \(\lambda^*_2\). Moreover, \(\Phi(\delta) = \delta^T \delta\) is a common Lyapunov function for collective dynamics of network.
Jointly Connected Graphs

Definition

\( G(k) \) Disconnected
Jointly Connected Graphs

Definition

$G(k + 1)$ Disconnected

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Jointly Connected Graphs

Definition

\[ \bigcup_{i=k,k+1} G(i) \] Jointly Connected over \([k, k + 1]\)
**Definition** Periodically Connected

Periodically connected with a period $N > 1$ if the union of all graphs over the interval $[j, jN)$ are jointly connected.

**Definition** Ultimately Connected

Ultimately connected if there exists an initial time $k_0$ such that over the infinite interval $[k_0, \infty)$ the union of all graphs over that interval are jointly connected.
Theorem 7

Consider the discrete-time consensus algorithm

\[ x_{k+1} = P_{s_k} x_k \]

with \( P_{s_k} \in \{ P_{s_1}, \ldots, P_{s_m} \} \). If the switching network is periodically connected then,

\[ \lim_{k \to \infty} x_k = \alpha 1. \]
**Theorem 8**

Consider the discrete-time consensus algorithm

\[ x_{k+1} = P_{s_k} x_k \]

with \( P_{s_k} \in \{ P_{s_1}, \ldots, P_{s_m} \} \). If the switching network is ultimately connected then a consensus is globally asymptotically reached.
Consider $N$ agents with state and measurement equations

$$\dot{x}_i = P_A x_i + P_B u_i$$
$$y_i = P_{C_1} x_i$$
$$z_{ij} = P_{C_2} (x_i - x_j), j \in N_i$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$

- Errors $z_{ij}$ synthesized into single signal

$$z_i = \frac{1}{|N_i|} \sum_{j \in N_i} z_{ij}$$
The control law $K$ is given as

$$
\dot{v}_i = K_A v_i + K_{B_1} y_i + K_{B_2} z_i \\
u_i = K_C v_i + K_{D_1} y_i + K_{D_2} z_i
$$

where $v_i$ is an internal state.
Formation Control System

Complete System Equations

\[
\begin{pmatrix}
\dot{x} \\
\dot{v}
\end{pmatrix}
= \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\begin{pmatrix}
x \\
v
\end{pmatrix}
\tag{E}
\]

where

\[
A_{11} = I_N \otimes (P_A + P_B P_{D1} P_{C1}) + (I_N \otimes P_B P_{D2} P_{C2})(L \otimes I_n)
\]

\[
A_{12} = I_N \otimes P_B K_C
\]

\[
A_{21} = I_N \otimes K_{B1} P_{C1} + (I_N \otimes K_{B2} P_{C2})(L \otimes I_n)
\]

\[
A_{22} = I_N \otimes K_A
\]

\(\otimes\) represent Kronecker product
Remarks

- Aim is to drive difference in states to zero → achieves rendezvous
- Relative positions achieved by putting in an offset into equations
- However, this does not affect stability analysis
Theorem 9

A local controller $K$ stabilizes the formation dynamics in (E) iff it simultaneously stabilizes the set of $N$ systems

\[
\dot{x}_i = P_A x_i + P_B u_i
\]
\[
y_i = P_{C_1} x_i
\]
\[
z_i = \lambda_i P_{C_2} (x_i - x_j)
\]

where $\lambda_i$ are the eigenvalues of $L$
Formation Control System

Proof of Stability Theorem

Let $T$ be Schur transformation of $L \Rightarrow U = T^{-1}LT$

$T \otimes I_n$ transforms $L \otimes I_n$ into $U \otimes I_n$

Define $\tilde{x} = (T \otimes I_n)x$ and $\tilde{v} = (T \otimes I_m)v$

Make use of identity

$$(I_N \otimes X)(Y \otimes I_s) = (Y \otimes I_r)(I_N \otimes X) = Y \otimes X$$

where $X$ is $r \times s$ matrix and $Y$ is $N \times N$ matrix
Formation Control System

Proof of Stability Theorem (cntd...)

\[
\begin{pmatrix}
\dot{\tilde{x}} \\
\dot{\tilde{v}}
\end{pmatrix} =
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\begin{pmatrix}
\tilde{x} \\
\tilde{v}
\end{pmatrix}
\]

where

\[
\begin{align*}
A_{11} &= I_N \otimes (P_A + P_B P_{D_1} P_{C_1}) + (I_N \otimes P_B P_{D_2} P_{C_2})(U \otimes I_n) \\
A_{12} &= I_N \otimes P_B P_{C} \\
A_{21} &= I_N \otimes K_{B_1} P_{C_1} + (I_N \otimes K_{B_2} P_{C_2})(U \otimes I_n) \\
A_{22} &= I_N \otimes K_A
\end{align*}
\]

Note: Elements are either block diagonal or block upper-triangular
The \( N \) diagonal subsystems can be written as

\[
\begin{align*}
\dot{x}_i &= (P_A + P_B K_{D_1} P_{C_1} + \lambda_i P_B K_{D_2} P_{C_2})\tilde{x}_i + P_B K_C \tilde{v}_i \\
\dot{\tilde{v}}_i &= (K_{B_1} P_{C_1} + \lambda_i K_{B_2} P_{C_2})\tilde{x}_i + K_A \tilde{v}_i
\end{align*}
\]

Equivalent to controller \( K \) stabilizing system in \( (F) \)
Formation Control System
Strategy for Controller Design

- Need to design a controller $K$ to stabilize

\[
\begin{align*}
\dot{x}_i &= P_A x_i + P_B u_i \\
y_i &= P_{C_1} x_i \\
z_i &= \lambda_i P_{C_2} (x_i - x_j)
\end{align*}
\]

- A prudent design strategy

1. close an inner loop around $y_i$ to stabilize vehicle dynamics
2. close an outer loop around $z_i$ to achieve desired performance
Theorem 10

Suppose $P$ is a SISO system. Then $K$ stabilizes the relative dynamics of formation if and only if the net encirclement of $-1/\lambda_i$ by Nyquist plot of $-K(s)P(s)$ is zero for all nonzero $\lambda_i$. 
Nyquist criterion states

"Number of CCW encirclements of $-1 + j0$ by forward loop $\lambda_i P(j\omega)K(j\omega) = Number of right half plane poles of $P(s)$" (which is zero by assumption)

Equivalent to number of encirclements of $-\lambda_i^{-1}$ by $P(j\omega)K(j\omega)$ being zero.
Background Theory Consensus Protocol Convergence Analysis Formation Control Applications

Formation Control System
Example and its Nyquist Plot

Example: Second order system with time delay $P(s) = \frac{e^{-s\tau}}{s^2}$
Formation Control System
Network Topologies and Corresponding Nyquist Locations

Sample Graphs, Spectra, and Nyquist Locations

<table>
<thead>
<tr>
<th>Graph Structure</th>
<th>Spectral Properties</th>
<th>Nyquist Location</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda(L) = {1 + \frac{1}{N-1}}$</td>
<td>$-1$</td>
</tr>
<tr>
<td></td>
<td>$\lambda(L) = {1}$</td>
<td>$-1$</td>
</tr>
<tr>
<td></td>
<td>$\lambda(L) \in [0, 2]$</td>
<td>$-1$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_i(L) = 1 - e^{\frac{2\pi i}{N}}$</td>
<td>$-1$</td>
</tr>
<tr>
<td></td>
<td>$\lambda(L) \ni 2$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>
Example: Second order system with time delay $P(s) = \frac{e^{-s\tau}}{s^2}$
Nonlinear consensus protocols
Consensus over random networks
Relaxed sufficiency conditions for convergence
Consensus in asynchronous time
Applications
Related to Consensus Problem

1. Synchronization of coupled oscillators
2. Flocking theory
3. Fast consensus in small worlds
4. Rendezvous in space
5. Distributed sensor fusion in sensor networks
6. Distributed formation control
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To Sum Up

Summary

- Consensus protocol for networks
- Convergence and performance of protocols
- Stability of distributed formation control